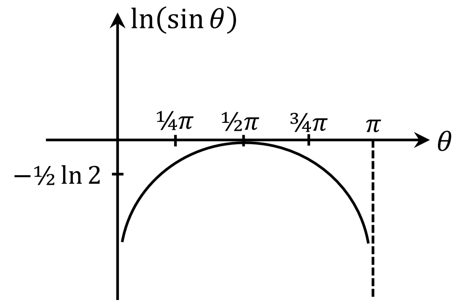
Problem 22) a) The function $\ln(\sin\theta)$ equals zero at $\theta = \pi/2$, and drops symmetrically to $-\infty$ as θ moves to the left toward zero, or to the right toward $\theta = \pi$. At $\theta = \pi/4$ and $\theta = 3\pi/4$, where $\sin\theta = \sqrt{2}/2$, we find $\ln(\sin\theta) = -½\ln 2$. A plot of the function over the interval $0 < \theta < \pi$ is shown below.



b) A change of variable from θ to $(½\pi - \theta)$ is all that is needed to demonstrate the equality of the areas under $\ln(\cos\theta)$ and $\ln(\sin\theta)$ over the interval $0 < \theta < \pi/2$, that is,

$$\int_0^{\pi/2} \ln(\cos\theta)\,\mathrm{d}\theta = -\int_{\pi/2}^{0} \ln[\cos(½\pi - \theta)]\,\mathrm{d}\theta = \int_0^{\pi/2} \ln(\sin\theta)\,\mathrm{d}\theta$$

To integrate $\ln(\cos\theta)$ from 0 to $\pi/2$, we write $\cos\theta$ as $½(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta})$, then expand the resulting integrand as follows:

$$\begin{aligned}
\int_0^{\pi/2} \ln(\cos\theta)\,\mathrm{d}\theta &= \int_0^{\pi/2} \ln[½(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta})]\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} [\ln(1 + e^{\mathrm{i}2\theta}) - \ln(2e^{\mathrm{i}\theta})]\mathrm{d}\theta \\
&= \int_0^{\pi/2} \Big\{ \sum_{n=1}^{\infty} [(-1)^{n+1}/n] e^{\mathrm{i}2n\theta} - (\ln 2 + \mathrm{i}\theta) \Big\} \mathrm{d}\theta \\
&= \sum_{n=1}^{\infty} [(-1)^{n+1}/n] \int_0^{\pi/2} e^{\mathrm{i}2n\theta}\mathrm{d}\theta - (\pi/2)\ln 2 - \mathrm{i}\int_0^{\pi/2} \theta\mathrm{d}\theta \\
&= \sum_{n=1}^{\infty} [(-1)^{n+1}(e^{\mathrm{i}n\pi} - 1)/(\mathrm{i}2n^2)] - (\pi/2)\ln 2 - \mathrm{i}(\pi^2/8) \\
&= \mathrm{i} \sum_{n=1,3,5,\cdots}^{\infty} (1/n^2) - (\pi/2)\ln 2 - \mathrm{i}(\pi^2/8).
\end{aligned}$$

Considering that $\sum_{n=1,3,5,\cdots}^{\infty} (1/n^2) = \pi^2/8$ (see chapter 4, Eq.(23)), we conclude that

$$\int_0^{\pi/2} \ln(\cos\theta)\,\mathrm{d}\theta = -(\pi/2)\ln 2.$$

c) Substituting the trigonometric identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ in the above integrand yields

$$\ln(\cos\theta) = \ln\sqrt{1 - \sin^2\theta} = ½\ln(1 - \sin^2\theta) = -\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta}{2n}.$$

It can be shown (using integration by parts) that $\int_0^{\pi/2} \sin^{2n}\theta\,\mathrm{d}\theta = \pi(2n-1)!!/[2(2n)!!]$. Consequently,

$$\int_0^{\pi/2} \ln(\cos\theta)\, \mathrm{d}\theta = -\sum_{n=1}^{\infty}(1/2n)\int_0^{\pi/2} \sin^{2n}\theta\, \mathrm{d}\theta = -(\pi/2)\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)(2n)!!}.$$

Given that the integral on the left-hand side of the above equation was shown in part (b) to be equal to $-(\pi/2)\ln 2$, we now have

$$\ln 2 = \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)(2n)!!} = \sum_{n=1}^{\infty}\frac{(2n)!}{2n[(2n)!!]^2} = \sum_{n=1}^{\infty}\frac{(2n)!}{2n(2^n n!)^2} = \sum_{n=1}^{\infty}\frac{1}{2^{2n+1}n}\binom{2n}{n}$$

$$= \frac{1}{4} + \frac{3}{32} + \frac{5}{96} + \frac{35}{1024} + \frac{63}{2560} + \frac{77}{4096} + \frac{429}{28672} + \cdots.$$

This is a legitimate series expansion for $\ln 2$, even though it converges slowly.